

**Analysis Of Coherent Random-
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WDMA/CDMA Multiplexing For
High-Capacity Optical Networks**

By

B. Ghaffari and E. Geraniotis

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ANALYSIS OF COHERENT RANDOM-CARRIER CDMA AND HYBRID WDMA/CDMA MULTIPLEXING FOR HIGH-CAPACITY OPTICAL NETWORKS

B. Ghaffari and E. Geraniotis

Department of Electrical Engineering
and Systems Research Center
University of Maryland
College Park, MD. 20742

ABSTRACT

In this paper we provide an accurate analysis of the performance of a random-carrier (RC) code-division multiple-access (CDMA) scheme recently introduced for use in high-capacity optical networks. According to this scheme coherent optical techniques are employed to exploit the huge bandwidth of single-mode optical fibers and are coupled with spread-spectrum direct-sequence modulation in order to mitigate the interference from other signals due to the frequency overlap caused by the instability of the carrier frequency of the laser, or to the mistakes in the frequency coordination and assignment.

The average bit error probability of this multiplexing scheme is evaluated by using the characteristic function of the other-user interference at the output of the matched optical filter. Both phase noise and thermal noise (AWGN) are taken into account in the computation. Both synchronous and asynchronous systems are analyzed in this context. The analysis is valid for any spreading gain and any number of interfering users and makes very limited use of approximations. The performance evaluation of RC CDMA establishes the potential advantage in employing hybrids of WDMA (wavelength-division multiple-access) and CDMA multiplexing to combat inter-carrier interference in dense WDMA systems.

I. Introduction

This paper is motivated by the recent work of [1], where a new random-carrier (RC) code-division multiplexing (CDMA) scheme was introduced for use in high-capacity optical networks. According to this scheme, coherent optical techniques are employed to exploit the huge bandwidth (tens of thousands of GHz) of single-mode optical fibers. The inherent instabilities of present-day semiconductor lasers are circumvented by coupling the optical multiple-access system, which is assumed to place randomly the modulated carriers in the available optical band, with CDMA. In particular, spread-spectrum direct-sequence modulation is employed in order to mitigate the interference from other signals due to the frequency overlap caused by the instability of the carrier frequency of the laser.

The transmitter/receiver model and a preliminary performance evaluation of this system were provided in [1]. For the sake of simplifying the analysis, several assumptions were made in [1] about the noise environment (for example, no thermal noise or phase noise were included) and the key parameters of the system (such as spreading gain, number of interfering users) were assumed infinitely large so that limiting theorems (e.g., the Central Limit Theorem) could be used. Furthermore, the accuracy of the approximations used was not justified for the range of the parameters of interest. In this paper, we extend and validate the work of [1] by providing an exact evaluation of the performance of the RC CDMA scheme without making all of the approximations and limiting assumptions made there.

The following assumptions were made in [1]: (i) the number of interfering users was taken to be infinite; (ii) the number of chips per bit (spreading gain) of the spread-spectrum modulation was assumed to be very large so that the Central Limit Theorem can be used; (iii) the error probability was not directly evaluated and the outage probability was calculated in terms of the signal-to-noise ratio, which required the signals at the output of the optical matched filter to be Gaussian in order to be valid; (iv) the time delays of the various (possibly asynchronous) users did not enter in the evaluation of the performance; and (iv) the effects of the phase noise and thermal noise on the system performance were not taken into account. Item (iv) pertains not only to the effects of the phase noise on the

single-user performance, but also to the effects of the phase noise on the CDMA system performance caused by its effect on the interference terms due to the other users. Although all of the above assumptions are relaxed in our paper, in the calculation of the characteristic function of the multiuser interference, which is essential for the approach followed in our paper, in one case the Central Limit Theorem is used and in the other case some other approximation on the phase signature sequence is made.

Relaxing the above assumptions for the system of [1] is an important contribution, necessary for validating the usefulness of CDMA in a variety of optical network configurations. Specifically, in some applications the number of interfering users is not large, because only a small number of users occupying adjacent frequency bands cause interference. Thus assumption (i) of the previous paragraph, which is critical for the analysis of [1], need to be relaxed.

The analysis presented in our paper is valid for an arbitrary number of interfering users and an arbitrary large number of chips per bit of the direct-sequence modulation; moreover, it takes into account the time delays of the interfering users, as well as the phase noise of the lasers (modeled as a Brownian motion process) and the thermal noise present at the receiver model (modeled as additive white Gaussian noise [AWGN]). The performance measure is the average probability of a bit error.

For the evaluation of the bit error probability, we use the characteristic-function method introduced in [2] for radio-frequency (RF) CDMA communication systems. The evaluation of error probability in [2] was carried out for arbitrary deterministic signature sequences; in [3] the results of [2] were extended to CDMA systems employing random signature sequences (i.e., mutually independent for different users i.i.d sequences that assume the values $+1$ and -1 with equal probability). Random signature sequences are a very useful model for (a) signature sequences, for which there is not much information available, or (b) for situations characterized by a large number of potential users and a small number of users that are active or are expected to cause interference. In that last case, one can not assign orthogonal sequences to all users and thus needs a large number of sequences with reasonable crosscorrelation properties. Random sequences make the performance evaluation of CDMA systems with very large populations feasible. In this

paper, we analyze optical RC CDMA systems with random signature sequences. In our analysis, BPSK modulation, as well as On-Off keying modulation is used to modulate the data bit stream, while MPSK modulation is employed for the signature sequence stream. Electro-optical phase modulators [4] make these features feasible.

We evaluate the average bit error probability by averaging over the data streams, signature sequences, carrier frequencies, phases of the interfering users, and time delays (for asynchronous systems). This is accomplished by computing the characteristic function of the interference due to the other users at the output of the optical filter matched to a particular signal. The computational complexity remains linear in the number of users and in the number of chips per bit (the spreading gain), as it was the case in [2] and [3]. The accuracy of this computational technique can be completely controlled by the user and is determined by the accuracy of the integration routines invoked; as the the number of points in the integration rule increases, the required computer CPU time increases. Any desirable accuracy can be attained via this technique.

Complete numerical results describing the performance of the RC CDMA system will be provided in the near future and will be reported in an expanded version of this report. The tradeoffs between the various system parameters (processing gain, signal-to-AWGN or signal-to-phase noise ratio, number of interfering users, and optical bandwidth) will be illustrated and interpreted. Besides the average error probability as a function of the system parameters, the maximum number of users that can be supported with this scheme at a given error probability will be provided. A comparison with the results of [1] will be carried out to examine the effect of the approximations and assumptions made in [1].

As a byproduct of our analysis, an exact expression for the error probability of a single user coherent optical system (employing a spreader/despreader) disturbed by phase noise and AWGN is derived. The performance of this system is also analyzed in detail in this report and numerical results will be provided in the second version of the report for several values of the parameters of interest.

The performance evaluation of RC CDMA will validate the potential advantage in employing hybrids of wavelength-division (WDMA) and code-division (CDMA) multiplexing in high capacity optical networks. A scheme based on the principle of combining the best

features of both FDMA and CDMA multiplexing schemes was proposed recently in [5]. This scheme uses primarily WDMA for providing multiple-access capability and employs CDMA for protection against laser-frequency instabilities and mistakes in the frequency coordination and assignment. This hybrid scheme shows great potential for application to high-capacity optical networks. Our analysis can be extended to this hybrid scheme. Finally, our analysis will enable us to compare the multiple-access capability of this hybrid WDMA/CDMA scheme to that of a pure WDMA scheme without CDMA protection.

This paper is organized as follows. In section *II*, the model and the receiver structure are described. In section *III*, the single user performance for the *BPSK* and *OOK* modulation is obtained. Section *IV* extends the analysis to the multiuser case in which K active users share a common optical channel. The average bit error probability for the intended user is obtained by using the characteristic functions of the interference and *AWGN*. This section also contains a subsection for the computation of the characteristic function of the other-user interference. Based on two different sets of assumptions, this function is obtained for both *BPSK* and *OOK* modulation. In section *V*, the *pdf* of a useful random variable which plays an important role in the analysis of both single user and multiuser systems, is estimated. Section *VI* extends the analysis to the asynchronous system, in which random delays are introduced to the signals of the interfering users.

II. Model

There are K users which share a common optical channel in a multiaccess fashion. The transmitted optical signal is denoted by $S(t)$ which is a complex signal in the form of

$$S(t) = \sum_{m=1}^K \sqrt{P} b_m(t) a_m(t) e^{i[\omega_m t + \theta_m(t)]} \quad (1)$$

where associated parameters are as follow

- P is the transmitted signal power of each user.
- $b_m(t)$ is the data stream of the m -th user given by

$$b_m(t) = \sum_{n=-\infty}^{\infty} b_n^{(m)} p(t - nT)$$

where $b_n^{(m)}$ denotes the n -th bit of the m -th user; $b_n^{(m)} \in \{-1, 1\}$ for BPSK modulation and $b_n^{(m)} \in \{0, 1\}$ for OOK modulation. $p(t)$ is a pulse of unit amplitude in $[0, T]$.

$a_m(t)$ is the addressing function or signature sequence stream used by user m . That is

$$a_m(t) = e^{i\phi_m(t)} = \sum_{n=-\infty}^{\infty} e^{i\phi_{mn}} h(t - nT_c)$$

where $h(t)$ is a pulse of unit amplitude in $[0, T_c]$ and $T_c = \frac{T}{N}$ is the chip duration where N is the number of chips per bit. ϕ_{mn} is a phase taking values in $[-\pi, \pi]$.

ω_m is the carrier on which the m -th signal is sent. This value is randomly chosen by the transmitter laser for (RC) CDMA, or is preassigned for hybrid WDMA/CDMA.

$\theta_m(t)$ is the phase noise associated with the m^{th} transmitter laser which is a Brownian motion process with Lorentzian bandwidth β . The mean of this process is zero and the variance is $2\pi\beta t$. At the k -th receiver, the optical signal $S(t)$ is first de-spread by $a_k^*(t)$ which is the complex conjugate of $a_k(t)$, and then homodyne-detected for the transmitted signal from user k (see Fig. 1). The output of the photodetector is

$$r(t) = \sqrt{P} b_k(t) e^{i\Delta\theta_k(t)} + \sum_m^I \sqrt{P} b_m(t) e^{i[\omega'_m t + \phi_m(t) - \phi_k(t) + \Delta\theta_m(t)]} + n(t) \quad (2)$$

where

$$\sum_m^I \triangleq \sum_{\substack{m=1 \\ m \neq k}}^K$$

$$\omega'_m \triangleq \omega_m - \omega_k$$

$$\Delta\theta_k(t) \triangleq \theta_k(t) - \theta_L(t)$$

$$\Delta\theta_m(t) \triangleq \theta_m(t) - \theta_L(t)$$

where $\theta_L(t)$ is the phase noise of the local laser and $n(t)$ is the complex *AWGN* process with double-sided spectral density $\frac{N_0}{2}$. The receiver used is a correlation receiver which is optimum for the single user case with no phase noise (see Fig. 2). The performance of

this suboptimum receiver in the presense of phase noise and AWGN is obtained for the single user and multiuser situations in the following sections.

III. Single User Analysis

In this section the performance of the system described in section II is evaluated in terms of BER for the single user case. The real part of the output of of the integrator is denoted by Y which is

$$Y = b_0^{(k)} X \sqrt{P} + \eta \sqrt{P} \quad (3)$$

where

$$X = \frac{1}{T} \int_0^T \cos[\Delta\theta_k(t)] dt \quad (4)$$

and η is a zero mean Gaussian random variable of variance $\frac{N_0}{2P_T}$. The probability of error P_e , for BPSK modulation is

$$P_e = \frac{1}{2} Pr \left[Y > \rho \sqrt{P} | b_0^{(k)} = -1 \right] + \frac{1}{2} Pr \left[Y < \rho \sqrt{P} | b_0^{(k)} = 1 \right] \quad (5)$$

where $\rho \sqrt{P}$ is the threshold. Upon substitution for Y from (3)

$$P_e = \frac{1}{2} Pr [\eta > \rho + X] + \frac{1}{2} Pr [\eta < \rho - X] \quad (6)$$

The random variable X takes values in $[-1, 1]$. P_e in (6) takes two possible forms depending on the values of X

$$P_e = \begin{cases} \frac{1}{2} - \frac{1}{2} Pr [\rho - X < \eta < \rho + X / X > 0] & w.p. \quad p^* \triangleq Pr [X > 0] \\ \frac{1}{2} + \frac{1}{2} Pr [\rho + X < \eta < \rho - X / X < 0] & w.p. \quad q^* \triangleq Pr [X < 0] \end{cases} \quad (7)$$

Where " $w.p.$ " means "with probability". Taking average of (7), we obtain

$$P_e = \frac{1}{2} + \frac{q^*}{2} Pr [\rho + X < \eta < \rho - X / X < 0] - \frac{p^*}{2} Pr [\rho - X < \eta < \rho + X / X > 0] \quad (8)$$

We now find the two probabilities in (8). The first one is

$$\begin{aligned} & \frac{1}{q^*} Pr [\rho + X < \eta < \rho - X, X < 0] \\ &= \frac{1}{q^*} \int_{-1}^0 Pr [\rho + X < \eta < \rho - X] f_X(x) dx \end{aligned} \quad (9)$$

where $f_X(\cdot)$ is the *pdf* of the random variable X . By using the functions $\Phi(\cdot)$ and $Q(\cdot)$ (9) takes the form

$$\frac{1}{q^*} \int_{-1}^0 \left[\Phi \left((\rho - X) \sqrt{\frac{2PT}{N_0}} \right) + Q \left((\rho + X) \sqrt{\frac{2PT}{N_0}} \right) - 1 \right] f_X(x) dx \quad (10)$$

where $Q(\cdot)$ and $\Phi(\cdot)$ are related to the standard normal distribution as follow

$$\begin{aligned} Q(\alpha) &= \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot dx \\ \Phi(\alpha) &= \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot dx \end{aligned}$$

Similarly, the second integral in (8) is

$$\frac{1}{p^*} \int_0^1 \left[1 - \Phi \left((\rho - X) \sqrt{\frac{2PT}{N_0}} \right) - Q \left((\rho + X) \sqrt{\frac{2PT}{N_0}} \right) \right] f_X(x) dx \quad (11)$$

Upon substitution of (10) and (11) in (8) the final result is obtained as

$$P_e = \frac{1}{2} \overline{Q \left((\rho + X) \sqrt{\frac{2PT}{N_0}} \right)} + \frac{1}{2} \overline{\Phi \left((\rho - X) \sqrt{\frac{2PT}{N_0}} \right)}. \quad (12)$$

The overlines in (12) indicate expectation with respect to the random variable X . Similarly, for *OOK* modulation, P_e is obtained as

$$P_e = \frac{1}{2} Q \left(\rho \sqrt{\frac{2PT}{N_0}} \right) + \frac{1}{2} \overline{\Phi \left((\rho - X) \sqrt{\frac{2PT}{N_0}} \right)}. \quad (13)$$

IV. Multiuser Analysis

A) BER Performance

In this section the performance of the system of section II is evaluated in terms of BER for the multiuser case. The output of the integrator V is

$$V = b_0^{(k)} \sqrt{P} x + \sqrt{P} \sum_m^I i_m + n \sqrt{P} \quad (14)$$

where

$$x = \frac{1}{T} \int_0^T e^{i\Delta\theta_k(t)} dt, \quad (15)$$

$$i_m = b_0^{(m)} \cdot \frac{1}{T} \int_0^T e^{i[\omega'_m t + \phi_m(t) - \phi_k(t) + \Delta\theta_m(t)]} dt, \quad (16)$$

and

$$n = \frac{1}{T\sqrt{P}} \int_0^T n(t) dt. \quad (17)$$

To calculate the integral in (16), we make some assumptions. First assume the phase addressing functions $\phi_m(t)$ and $\phi_k(t)$ in the chip interval $((n-1)T_c, nT_c]$ take the values ϕ_{mn} and ϕ_{kn} in $[-\pi, \pi]$, respectively. Second, for large values of N , it is reasonable to assume

$$\Delta\theta_m(t) = \Delta\theta_m(nT_c) \triangleq \theta_{mn} \quad (n-1)T_c < t \leq nT_c \quad (18)$$

where θ_{mn} is a zero mean Gaussian random variable of variance $4\pi\beta nT_c$. Under these assumptions

$$i_m = \frac{b_0^{(m)}}{N} \text{sinc}\left(\frac{\omega'_m T_c}{2}\right) \sum_{n=1}^N e^{i[\phi_{mn} - \phi_{kn} + \theta_{mn} + \omega'_m(n-1/2)T_c]}. \quad (19)$$

By taking the real part of V in (14)

$$Y = \left(b_0^{(k)} \cdot X + \sum_m^I I_m + \eta \right) \sqrt{P} \quad (20)$$

where X and η are defined in (3) and (4) , and I_m is

$$I_m = \frac{b_0^{(m)}}{N} \text{sinc} \left(\frac{\omega'_m T_c}{2} \right) \sum_{n=1}^N \cos(X_{mn}) \quad (21)$$

where

$$X_{mn} = \langle \phi_{mn} - \phi_{kn} + \theta_{mn} + \omega'_m (n - 1/2) T_c \rangle . \quad (22)$$

In (22) $\langle \cdot \rangle$ represents $[\cdot] \bmod 2\pi$. To evaluate the bit error probability, let rewrite (20) as

$$Y = \left(b_0^{(k)} \cdot X + I + \eta \right) \sqrt{P} \quad (23)$$

where

$$I = \sum_m^I I_m . \quad (24)$$

The probability of error P_e , for BPSK modulation, can be obtained from (5) as follows. If we replace “ η ” with “ $I + \eta$ ” equations (6),(7) and (8) are still valid for multiuser case. The first probability in (8) is

$$\begin{aligned} & \frac{1}{q^*} \Pr [\rho + X < I + \eta < \rho - X, X < 0] \\ &= \frac{1}{q^*} \int_{-1}^0 \Pr [\rho + X < I + \eta < \rho - X] f_X(x) dx \end{aligned} \quad (25)$$

Upon substitution for $\Pr [\cdot]$ in (25) by an integral,

$$\frac{1}{q^*} \int_{-1}^0 dx f_X(x) \cdot \int_{\rho+x}^{\rho-x} f_{I+\eta}(y) dy, \quad (26)$$

where $f_{I+\eta}(\cdot)$ is the pdf of $I + \eta$. Since the pdf $f_{I+\eta}(\cdot)$ is a real function, it is easy to show that

$$f_{I+\eta}(y) = \frac{1}{\pi} \int_0^\infty [\text{Re} \{ \Phi_{I+\eta}(u) \} \cdot \cos(yu) + \text{Im} \{ \phi_{I+\eta}(u) \} \sin(yu)] du \quad (27)$$

where $Re\{\cdot\}$ and $Im\{\cdot\}$ denote the real and imaginary parts, respectively, of their complex arguments. Considering that I and η are independent and that $\Phi_I(u)$ turned out to be real (see part B of this section),

$$f_{I+\eta}(y) = \frac{1}{\pi} \int_0^\infty \Phi_I(u) \Phi_\eta(u) \cos(yu) du \quad (28)$$

Substituting (28) in (26) and performing the integration $\int_{\rho+x}^{\rho-x}$, the first probability in (8) is

$$-\frac{2}{\pi q^*} \int_0^\infty \Phi_I(u) \Phi_\eta(u) \cos(\rho u) \left[\int_{-1}^0 f_X(x) \sin(ux) dx \right] \frac{du}{u} \quad (29)$$

Using the same procedure, the second probability in (8) is

$$\frac{2}{\pi p^*} \int_0^\infty \Phi_I(u) \Phi_\eta(u) \cos(\rho u) \left[\int_0^1 f_X(x) \sin(ux) dx \right] \frac{du}{u} \quad (30)$$

Combining (29) and (30) in (8), we obtain

$$P_e = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \Phi_I(u) \Phi_\eta(u) \cos(\rho u) \frac{\overline{\sin(uX)}}{u} du \quad (31)$$

where

$$\overline{\sin(uX)} = \int_{-1}^1 f_X(x) \sin(ux) dx.$$

To evaluate P_e for *OOK* modulation, we follow the same method as before and obtain

$$P_e = \frac{1}{2} - \frac{1}{2\pi} \int_0^\infty \Phi_I(u) \Phi_\eta(u) \cdot \frac{\sin(\rho u) - \overline{\sin(\rho - X)u}}{u} \cdot du \quad (32)$$

In order to put (31) in a more meaningful format and also to facilitate the computations, we rewrite (31) as

$$\begin{aligned} P_e &= \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \Phi_\eta(u) \cos(\rho u) \frac{\overline{\sin(uX)}}{u} du \\ &\quad + \frac{1}{\pi} \int_0^\infty (1 - \Phi_I(u)) \Phi_\eta(u) \cos(\rho u) \frac{\overline{\sin(uX)}}{u} du. \end{aligned} \quad (33)$$

The third term in (33) includes the contribution of the other-user interference in BER. The first two terms are single user contributions. Therefore, by using (13) in (33)

$$P_e = \frac{1}{2}Q\left((\rho + X)\sqrt{\frac{2PT}{N_0}}\right) + \frac{1}{2}\Phi\left((\rho - X)\sqrt{\frac{2PT}{N_0}}\right) + \frac{1}{\pi}\int_0^\infty (1 - \Phi_I(u))\Phi_\eta(u)\cos(\rho u)\frac{\overline{\sin(uX)}}{u}du. \quad (34)$$

Similarly for *OOK* modulation, P_e is

$$P_e = \frac{1}{2}Q\left(P\sqrt{\frac{2PT}{N_0}}\right) + \frac{1}{2}\Phi\left((\rho - X)\sqrt{\frac{2PT}{N_0}}\right) + \frac{1}{2\pi}\int_0^\infty (1 - \Phi_I(u))\Phi_\eta(u)\frac{\sin(\rho u) - \overline{\sin(\rho - X)u}}{u} \cdot du \quad (35)$$

where

$$\Phi_\eta(u) = \exp\left(-\frac{N_0}{4PT}u^2\right) \quad (36)$$

B) Characteristic Function Computation

The computation of P_e in (34) and (35) requires the characteristic function of the multiuser interference, $\Phi_I(u)$. This is defined as

$$\Phi_I(u) = E[e^{iuI}] = E\left[e^{iu\sum_m' \frac{b_0^{(m)}}{N}\text{sinc}(\frac{\omega_m' T_c}{2})\sum_{n=1}^N \cos(X_{mn})}\right] \quad (37)$$

where X_{mn} is defined in (22). Here, two approaches are followed for obtaining this characteristic function.

According to the first approach, it is assumed that the carrier frequencies of the users are randomly and *uniformly* distributed in a bandwidth of W . The choice of the phase signature sequences distribution is *arbitrary*. This means that the sequence $\{\phi_{mn}\}$ takes values in $[-\pi, \pi]$ with *arbitrary* distribution.

In the second approach, it is assumed that the phase signature sequences are *continuous* and *uniformly* distributed in $[-\pi, \pi]$. This approximates the case in which $\{\phi_{mn}\}$

takes M distinct levels in $[-\pi, \pi]$ with probability $\frac{1}{M}$ each, where the number of levels M is large. In this case, the choice of the carrier frequencies distributions is *arbitrary*.

1. Assumption of Uniform Carriers

Let X_{mn} in (37) be

$$X_{mn} = \langle \gamma_{mn} + \beta_{mn} \rangle \quad (38)$$

where

$$\gamma_{mn} = \phi_{mn} - \phi_{kn} + \theta_{mn} \quad (39)$$

$$\beta_{mn} = \omega'_m(n - 1/2)T_c \quad (40)$$

$\{\omega'_m\}$ is assumed to be iid and uniformly distributed in a bandwidth W as big as 10 THZ . Therefore, $\{\beta_{mn}\}$ for fixed n are iid and uniformly distributed in a bandwidth of $(n - 1/2)W/(RN)$, where R is the data rate. For typical values of R and N this value is still very large and it is reasonable to consider $\{\langle \beta_{mn} \rangle\}$ as iid random variables uniformly distributed in $[-\pi, \pi]$ (see Appendix B). According to Appendix A , $\{X_{mn}\}$ with respect to m are iid and uniformly distributed in $[-\pi, \pi]$. Therefore, (37) is expressed as

$$\Phi_I(u) = \prod_m' E \left[e^{iu \frac{b_0^{(m)}}{N} \text{sinc}(\frac{\omega'_m T_c}{2}) \sum_{n=1}^N \cos(X_{mn})} \right]. \quad (41)$$

This is independent of the index m . Hence

$$\Phi_I(u) = \left\{ E \left[e^{iu \frac{b_0}{N} \text{sinc}(\frac{\omega T_c}{2}) \sum_{n=1}^N \cos(X_n)} \right] \right\}^{(K-1)} \quad (42)$$

Assuming $Y_n \triangleq \cos(X_n)$, it is easy to show that the sequence $\{Y_n\}$ has zero mean and zero correlation. Therefore, this sequence is a " ρ - mixing" (See [6] and [7]) with $\rho(n) = 0$.

By using *CLT* for large N , the term $\frac{1}{\sqrt{N}} \sum_{n=1}^N \cos(X_n)$ in (42) can be replaced by ζ , where ζ is a Gaussian random variable of zero mean and variance $\frac{1}{2}$. Therefore, (42) becomes

$$\Phi_I(u) = \left\{ E \left[e^{iu \frac{b_0}{\sqrt{N}} \text{sinc}(\frac{\omega T_c}{2}) \zeta} \right] \right\}^{(K-1)}. \quad (43)$$

The expression above is the characteristic function of a Gaussian random variable. Therefore,

$$\Phi_I(u) = \left\{ E \left[e^{-u^2 \frac{b_0^2}{4N} \left(\text{sinc}\left(\frac{\omega T_c}{2}\right) \right)^2} \right] \right\}^{(K-1)} \quad (44)$$

where the expectation is with respect to b_0 and ω . For BPSK, $b_0 \in \{-1, 1\}$, consequently we obtain

$$\Phi_I(u) = \left\{ \exp \left[-\frac{u^2}{4N} \left(\text{sinc}\left(\frac{\omega T_c}{2}\right) \right)^2 \right] \right\}^{(K-1)}. \quad (45)$$

For OOK, $b_0 \in \{0, 1\}$, consequently we obtain

$$\Phi_I(u) = \left\{ \frac{1}{2} + \frac{1}{2} \exp \left[-\frac{u^2}{4N} \left(\text{sinc}\left(\frac{\omega T_c}{2}\right) \right)^2 \right] \right\}^{(K-1)}. \quad (46)$$

2. Assumption of Uniform Phase Signature Sequence

For the case in which the signature sequence phases are uniformly distributed in a set of equally spaced discrete levels, if the number of levels in this set is reasonably large, this discrete uniform distribution is approximated with a continuous uniform phase in $[-\pi, \pi]$.

Let us express the characteristic function of the interference I as

$$\Phi_I(u) = E_{\bar{\omega}'} E_{b_0} E_{\bar{X}} [e^{iuI}] \quad (47)$$

Where I is given by (24) and (21). $E_{\bar{X}}$ is the expectation with respect to the $N(K-1)$ dimensional vector in $\{X_{mn}\}$. E_{b_0} is the expectation with respect to the $K-1$ dimensional vector in $\{b_0^{(m)}\}$. Finally, $E_{\bar{\omega}'}$ is the expectation with respect to the $K-1$ dimensional vector $\bar{\omega}' = (\omega'_1, \dots, \omega'_K)$. Since $\{\phi_{mn}\}$ in (22) are iid and uniformly distributed in $[-\pi, \pi]$ for all m and n , based on the result established in Appendix A, $\{X_{mn}\}$ are also iid and uniform in $[-\pi, \pi]$ for all m and n . Therefore,

$$\Phi_I(u) = \prod_m^I E_{\omega'_m} \left\{ E_{b_0^{(m)}} \left\{ \prod_{n=1}^N E_{X_{mn}} \left[e^{i \frac{u}{N} b_0^{(m)} \cdot \text{sinc}\left(\frac{\omega'_m T_c}{2}\right) \cos(X_{mn})} \right] \right\} \right\}. \quad (48)$$

We use the identity

$$\begin{aligned} E_{X_{mn}} \left[e^{iu \cos(X_{mn})} \right] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iu \cos x} \cdot dx = \\ &= \frac{2}{\pi} \int_0^{\pi/2} \cos(u \cos x) dx = J_0(u) \end{aligned} \quad (49)$$

where $J_0(\cdot)$ is the Bessel function of the first kind. Using (49), the inner expectation in (48) becomes $J_0 \left(b_0^{(m)} \cdot \frac{u}{N} \cdot \text{sinc} \left(\frac{\omega'_m T_c}{2} \right) \right)$, which is independent of n . Also using the fact that $J_0(\alpha) = J_0(-\alpha)$, we obtain for BPSK

$$E_{b_0^{(m)}} \left\{ \prod_{n=1}^N E_{X_{mn}} [\cdot] \right\} = \left[J_0 \left(\frac{u}{N} \text{sinc} \left(\frac{\omega'_m T_c}{2} \right) \right) \right]^N. \quad (50)$$

Moreover, assuming that the sequence $\{\omega_m\}$ is *iid*, (50) will be independent of the index m and (48) becomes

$$\Phi_I(u) = \left\{ \overline{\left[J_0 \left(\frac{u}{N} \text{sinc} \left(\frac{\omega T_c}{2} \right) \right) \right]^N} \right\}^{k-1} \quad (51)$$

where the overline is the expectation with respect to the generic ω scattered in the bandwidth of the channel. Similarly, for *OOK* modulation, $\Phi_I(u)$ is obtained as

$$\Phi_I(u) = \left\{ \frac{1}{2} + \frac{1}{2} \overline{\left[J_0 \left(\frac{u}{N} \text{sinc} \left(\frac{\omega T_c}{2} \right) \right) \right]^N} \right\}^{k-1}. \quad (52)$$

V. PDF of X

The pdf of X which is defined in (4), depends on βT ; where 2β is the Lorentzian bandwidth of the Brownian motion process $\Delta\theta_k(t)$. In order to obtain this pdf through Monte Carlo simulation, (4) is written as

$$X = \frac{1}{T} \sum_{n=1}^F \int_{(n-1)\delta}^{n\delta} \cos(\Delta\theta_k(t)) dt \quad (53)$$

where F is the number of divisions of T into smaller portions of size δ , where $\delta = \frac{T}{F}$. For F large enough, let

$$\Delta\theta_k(t) = \Delta\theta_k(n\delta) \triangleq \theta_n \quad (n-1)\delta < t \leq n\delta \quad (54)$$

Upon substitution of (54), (53) becomes

$$X = \frac{1}{F} \sum_{n=1}^F \cos(\theta_n) \quad (55)$$

where $\{\theta_n\}$ is a sequence of zero mean Gaussian random variables of variance $4\pi\beta n\delta$ and correlation $4\pi\beta\delta \min(m, n)$. The statistics of X is estimated through the generation of the random sequence $\{\theta_n\}$ in a computer by using (55). The accuracy of this estimate is limited by the statistical fluctuations in this simulation. More specifically, the accuracy of the estimate probability of a quantile q , obtained with M independent simulation trials, is approximately $\sqrt{\frac{1}{Mq}}$. For more details, the study of a similar problem in [8] is helpful.

In order to generate $\{\theta_n\}$, the iteration below will provide the required properties for this sequence.

$$\begin{cases} \theta_1 = \nu_1 \\ \theta_n = \theta_{n-1} + \nu_n \quad 2 \leq n \leq F \end{cases} \quad (56)$$

where $\{\nu_n\}$ is a sequence of iid zero mean Gaussian random variables of variance $4\pi\beta\delta$. It is easy to show that the sequence $\{\theta_n\}$ has the required properties.

VI. Asynchronous System

In this section of this paper, the model and the analysis which was developed for the synchronous system in the previous sections, is extended to the asynchronous case. The transmitted optical signal; $S(t)$, is given in (1), where t is replaced by $t - \tau_m$. The time delay τ_m is considered to be a uniformly distributed random variable in $[0, T]$ which represents the m -th user's time delay. At the k -th receiver, the matched filter is synchronized with the k -th signal, i.e. $\tau_k = 0$. Therefore, equation (23) is still valid and the average probability of error is given in (34) and (35) for *BPSK* and *OOK* modulation, respectively. Evaluation

of the characteristic function of the multiuser interference for the asynchronous case follows next. Let us write τ_m as

$$\tau_m = \ell_m T_c + \tau'_m \quad (57)$$

where

$$\ell_m = \lfloor \frac{\tau_m}{T_c} \rfloor$$

The counterpart of (16) becomes

$$i_m = \frac{1}{T} \int_0^T b_m(t - \tau_m) e^{i[\omega'_m t - \omega'_m \tau_m + \phi_m(t - \tau_m) - \phi_k(t) + \Delta \theta_m(t)]} \quad (58)$$

The integral in (58) can be written as

$$\int_0^T = \int_0^{\tau_m} + \int_{\tau_m}^T \quad (59)$$

where

$$\int_0^{\tau_m} = \sum_{n=1}^{\ell_m+1} \int_{(n-1)T_c}^{(n-1)T_c + \tau'_m} + \sum_{n=1}^{\ell_m} \int_{(n-1)T_c + \tau'_m}^{nT_c} \quad (60)$$

$$\int_{\tau_m}^T = \sum_{n=\ell_m+2}^N \int_{(n-1)T_c}^{(n-1)T_c + \tau'_m} + \sum_{n=\ell_m+1}^N \int_{(n-1)T_c + \tau'_m}^{nT_c} \quad (61)$$

By using (59), (60) and (61) in (58), i_m becomes

$$i_m = \frac{1}{T} \sum_{n=1}^N \left[e_{mn}^+ \int_{(n-1)T_c}^{(n-1)T_c + \tau'_m} e^{ig_{mn}^+} \cdot dt + e_{mn}^- \int_{(n-1)T_c + \tau'_m}^{nT_c} e^{ig_{mn}^-} \cdot dt \right] \quad (62)$$

where

$$g_{mn}^+ \triangleq \omega'_m t - \omega'_m \tau_m + \phi_{m(n-1)} - \phi_{kn} + \theta_{mn} \quad (63)$$

$$g_{mn}^- \triangleq \omega'_m t - \omega'_m \tau_m + \phi_{mn} - \phi_{kn} + \theta_{mn} \quad (64)$$

$$e_{mn}^+ \triangleq b_{-1}^{(m)} \cdot 1_{[1, \ell_m+1]}(n) + b_0^{(m)} \cdot 1_{[\ell_m+2, N]}(n) \quad (65)$$

$$e_{mn}^- \triangleq b_{-1}^{(m)} \cdot 1_{[1, \ell_m]}(n) + b_0^{(m)} \cdot 1_{[\ell_m+1, N]}(n) \quad (66)$$

$$1_{[i, j]}(n) \triangleq \begin{cases} 1 & i \leq n \leq j \\ 0 & \text{other} \end{cases} \quad (67)$$

The counterpart of (21) is obtained as

$$I_m = R_e\{i_m\} = \frac{1}{T} \sum_{n=1}^N [e_{mn}^+ \alpha_m^+ \cos X_{mn}^+ + e_{mn}^- \alpha_m^- \cos X_{mn}^-] \quad (68)$$

where

$$\alpha_m^+ \triangleq \tau_m' \text{sinc} \left(\frac{\omega_m' \tau_m'}{2} \right) \quad (69)$$

$$\alpha_m^- \triangleq (T_c - \tau_m') \text{sinc} \left[\frac{\omega_m' (T_c - \tau_m')}{2} \right] \quad (70)$$

$$X_{mn}^+ \triangleq < \omega_m' (n-1) T_c + \frac{\omega_m' \tau_m'}{2} - \omega_m' \tau_m + \phi_{m(n-1)} - \phi_{kn} + \theta_{mn} > \quad (71)$$

$$X_{mn}^- \triangleq < \omega_m' (n-1/2) T_c + \frac{\omega_m' \tau_m'}{2} - \omega_m' \tau_m + \phi_{mn} - \phi_{kn} + \theta_{mn} > \quad (72)$$

At this point, similar to the synchronous case, we make two assumptions as follows.

1. Assumption of Uniform Carriers

As before, we observe that, in this case $\{X_{mn}^+\}$ and $\{X_{mn}^-\}$ with respect to m are iid and uniformly distributed in $[-\pi, \pi]$. Therefore,

$$\Phi_I(u) = \prod_m' E \left[e^{j \frac{u}{T}} (\alpha_m^+ \sum_{n=1}^N e_{mn}^+ \cos X_{mn}^+ + \alpha_m^- \sum_{n=1}^N e_{mn}^- \cos X_{mn}^-) \right] \quad (73)$$

This is independent of index m . Hence

$$\Phi_I(u) = \left\{ E \left[e^{j \frac{u}{T}} (\alpha^+ \sum_{n=1}^N e_n^+ \cos X_n^+ + \alpha^- \sum_{n=1}^N e_n^- \cos X_n^-) \right] \right\}^{(K-1)} \quad (74)$$

The sequences $\{e_n^+ \cos X_n^+\}$ and $\{e_n^- \cos X_n^-\}$ are “ ρ -mixing” sequences. Therefore

$$\Phi_I(u) = \left\{ E \left[e^{j \frac{u \sqrt{N}}{T}} (\alpha^+ \eta^+ + \alpha^- \eta^-) \right] \right\}^{(K-1)} \quad (75)$$

where η^+ and η^- are zero mean Gaussian random variables of variance $\frac{1}{2}$ for *BPSK* and $\frac{1}{4}$ for *OOK* modulation. Moreover, since η^+ and η^- are uncorrelated, they are independent. Therefore

$$\Phi_I(u) = \left\{ E \left[e^{j \frac{u}{T} \sqrt{N} \cdot \eta} \right] \right\}^{(K-1)} \quad (76)$$

where η is a conditionally zero mean Gaussian random variable of variance $\frac{1}{2} [(\alpha^+)^2 + (\alpha^-)^2]$ for *BPSK* and $\frac{1}{4} [(\alpha^+)^2 + (\alpha^-)^2]$ for *OOK* modulation. Finally, for *BPSK*

$$\Phi_I(u) = \left\{ \exp \left\{ -\frac{u^2}{4N} \left[\left(1 - \frac{\tau}{T_c}\right)^2 \text{sinc}^2(\omega(T_c - \tau)/2) + \left(\frac{\tau}{T_c}\right)^2 \text{sinc}^2\left(\frac{\omega\tau}{2}\right) \right] \right\} \right\}^{(K-1)} \quad (77)$$

where the expectation is with respect to ω and τ . As before ω is uniformly distributed in the bandwidth W and τ is uniformly distributed in $[0, T_c]$. For *OOK* modulation, only $4N$ should be replaced by $8N$ in (77).

2. Assumption of Uniform Phase Signature Sequence

Since the two sequences $\{\phi_{m(n-1)}\}$ in (71) and $\{\phi_{mn}\}$ in (72) are iid and uniformly distributed in $[-\pi, \pi]$ for all m and n , then, the two sequences $\{X_{mn}^+\}$ and $\{X_{mn}^-\}$ are also iid and uniformly distributed in $[-\pi, \pi]$ for all m and n . This validates equation (73) for this case. Moreover, the two sequences $\{X_{mn}^+\}$ and $\{X_{mn}^-\}$ are independent of each other. To prove this we need to show that the two random variables X_{mn}^+ and X_{ij}^- are independent for all m, n, i and j . For all cases, except the case in which $i = m$ and $j = n - 1$, the iid and uniform phase sequences $\{\phi_{m(n-1)}\}$ in (71) and $\{\phi_{ij}\}$ in (72) provide the proof in a straightforward manner. For the case $m = i$ and $j = n - 1$, X_{mn}^+ and X_{ij}^- are still independent, because ϕ_{kn} and $\phi_{k(n-1)}$ are independent and uniform for this case. Using these facts

$$\Phi_I(u) = \prod_m' E_m \left\{ \prod_{n=1}^N E_{X_{mn}^+} \left[e^{\frac{j u}{T} \alpha_m^+ e_{mn}^+ \cos X_{mn}^+} \right] \cdot E_{X_{mn}^-} \left[e^{\frac{j u}{T} \alpha_m^- e_{mn}^- \cos X_{mn}^-} \right] \right\} \quad (78)$$

where the expectation E_m is with respect to the all random variables with index m . Upon

substitution from (49)

$$\Phi_I(u) = \prod_m' E_m \left\{ \prod_{n=1}^N J_0 \left(\frac{u}{T} \alpha_m^+ e_{mn}^+ \right) J_0 \left(\frac{u}{T} \alpha_m^- e_{mn}^- \right) \right\} \quad (79)$$

For *BPSK* modulation e_{mn}^+ and e_{mn}^- belong to $\{-1, 1\}$. Therefore, the inner product is independent of n and (79) becomes

$$\Phi_I(u) = \prod_m' E_m \left[J_0 \left(\frac{u}{T} \alpha_m^+ \right) J_0 \left(\frac{u}{T} \alpha_m^- \right) \right]^N \quad (80)$$

Moreover, assumming that the sequences $\{\omega_m\}$ and $\{\tau_m\}$ are iid, the expectation above will be independent of the index m and finally, the characteristic function of the multiuser interference for *BPSK* modulation is obtained as

$$\Phi_I(u) = \left\{ \left[J_0 \left(\frac{u}{N} \cdot \frac{\tau}{T_c} \text{sinc} \left[\frac{\omega\tau}{2} \right] \right) J_0 \left(\frac{u}{N} \left(1 - \frac{\tau}{T_c} \right) \text{sinc} \left[\frac{\omega(T_c - \tau)}{2} \right] \right) \right]^N \right\}^{(K-1)} \quad (81)$$

where the expectation in (81) is with respect to ω and τ , which are distributed in $[-W/2, W/2]$ and $[0, T_c]$, respectively.

The evaluation of the characteristic function of the multiuser interference for *OOK* in this case, is more tedious than for *BPSK*. We know that ℓ_m is a uniform discrete random variable which takes values in $\{0, 1, \dots, N-1\}$. By assuming

$$c_m \triangleq J_0 \left(\frac{u}{T} \alpha_m^+ b_{-1}^{(m)} \right) J_0 \left(\frac{u}{T} \alpha_m^- b_{-1}^{(m)} \right) \quad (82)$$

$$d_m \triangleq J_0 \left(\frac{u}{T} \alpha_m^+ b_0^{(m)} \right) J_0 \left(\frac{u}{T} \alpha_m^- b_0^{(m)} \right) \quad (83)$$

$$v_m \triangleq J_0 \left(\frac{u}{T} \alpha_m^+ b_{-1}^{(m)} \right) J_0 \left(\frac{u}{T} \alpha_m^- b_0^{(m)} \right) \quad (84)$$

(79) becomes

$$\Phi_I(u) = \prod_m' E_m \left\{ \left[\prod_{n=1}^{\ell_m} c_m \right] \left[\prod_{n=\ell_m+2}^N d_m \right] v_m \right\}. \quad (85)$$

Since c_m , d_m and v_m are independent of n , then

$$\Phi_I(u) = \prod_m' E_m \{ v_m c_m^{\ell_m} d_m^{N-\ell_m-1} \}. \quad (86)$$

Performing the expectation above with respect to ℓ_m , yields

$$\Phi_I(u) = \prod_m' E_m \left\{ \frac{v_m}{N} \sum_{i=0}^{N-1} c_m^i d_m^{N-1-i} \right\}. \quad (87)$$

By taking expectation with respect to $(b_{-1}^{(m)}, b_0^{(m)})$ and rearranging the terms, the final result is

$$\begin{aligned} \Phi_I(u) = & \left\{ \frac{1}{4} + \frac{1}{4} \overline{[R(\omega, \tau) R(\omega, T_c - \tau)]^N} \right. \\ & \left. + \frac{1}{4N} \left[\overline{[R(\omega, \tau) + R(\omega, T_c - \tau)] \frac{1 - [R(\omega, \tau) R(\omega, T_c - \tau)]^N}{[R(\omega, \tau) R(\omega, T_c - \tau)]}} \right] \right\}^{(K-1)} \end{aligned} \quad (88)$$

where

$$R(\omega) = \frac{u}{T} \tau \text{sinc}\left(\frac{\omega \tau}{2}\right). \quad (89)$$

Appendix A

In this appendix , we show that if a sequence of *iid* random variables, which are *uniformly* distributed in $[-\pi, \pi]$, are added to any other sequence of random variables with *arbitrary* distribution, the resultant sequence *mod.2 π* is also *iid* and *uniformly* distributed in $[-\pi, \pi]$. To show this, let

$$X_n = \langle \phi_n + \lambda_n \rangle \quad (A1)$$

where $\{\phi_n\}$ is assumed to be iid and uniform in $[-\pi, \pi]$ for all n , and $\{\lambda_n\}$ is a sequence of random variables with arbitrary distribution. Here we establish two claims:

Claim 1: X_n and ϕ_n have the same distribution.

Proof : It is a known fact that

$$X_n | \text{given } \lambda_n \stackrel{d}{=} \phi_n \quad (A2)$$

The pdf of X_n is obtained as

$$f_{X_n}(x) = \int_{\lambda} f_{X_n|\lambda_n}(x|\lambda) f_{\lambda_n}(\lambda) d\lambda \quad (A3)$$

By usin (A2), (A3) becomes

$$f_{X_n}(x) = \int_{\lambda} f_{\phi_n}(x) f_{\lambda_n}(\lambda) d\lambda = f_{\phi_n}(x). \quad (A4)$$

Claim 2: $\{X_n\}$ are independent for all n.

Proof : We show that any two random variables X_n and X_l ($n \neq l$) are independent. To show this, it suffices to show that for any function $g(\cdot)$

$$E[g(X_n)g(X_l)] = E[g(X_n)]E[g(X_l)] \quad (A5)$$

Given λ_n and λ_l in (A1), the first side of (A5) conditioned on these values is

$$E[g(\langle \phi_n + \lambda_n \rangle).g(\langle \phi_l + \lambda_l \rangle) | \lambda_n, \lambda_l]. \quad (A6)$$

Since ϕ_n and ϕ_l are independent, (A6) becomes

$$E[g(< \phi_n + \lambda_n >)|\lambda_n] \cdot E[g(< \phi_l + \lambda_l >)|\lambda_l] \quad (A7)$$

or

$$E[g(X_n)|\lambda_n] \cdot E[g(X_l)|\lambda_l]. \quad (A8)$$

From Claim 1:

$$X_n \stackrel{d}{=} X_n | \text{given } \lambda_n, \quad (A9)$$

$$X_l \stackrel{d}{=} X_l | \text{given } \lambda_l.$$

Therefore, (A8) is

$$E[g(X_n)] \cdot E[g(X_l)]. \quad (A10)$$

This means that

$$E[g(X_n)g(X_l)|\lambda_n, \lambda_l] = E[g(X_n)] \cdot E[g(X_l)]. \quad (A11)$$

Taking expectation respect to the λ_n and λ_l gives (A5) and this completes this proof.

Appendix B

In this appendix, the distribution of a random variable which is obtained as the *mod*. 2π of a uniform random variable is derived. Let assume X to be a random variable which is uniformly distributed in the bandwidth $[0, W]$. We are interested to know the distribution of the random variable Y which is defined as

$$Y = < X > . \quad (B1)$$

The random variable Y is expressed in terms of some disjoint intervals as follows

$$Y = \begin{cases} X - 2n\pi & \text{if } 2n\pi < X < 2(n+1)\pi; \\ X - 2M\pi & \text{if } 2M\pi < X < W \end{cases} \quad 0 \leq n < M \quad (B2)$$

where

$$M = \lfloor W/2\pi \rfloor$$

For $\alpha \in [0, 2\pi]$, the event $(Y \leq \alpha)$ is expressed as

$$\bigcup_{n=0}^{M-1} (X - 2n\pi \leq \alpha, \quad 2n\pi < X < 2(n+1)\pi) \bigcup (X - 2M\pi \leq \alpha, \quad 2n\pi < X < W) \quad (B3)$$

The probability of (B3) is the summation of the probabilities of the individual events in (B3). By using the conditional probability and after some simplifications, $F_Y(\alpha)$; the cumulative distribution function of Y , is

$$F_Y(\alpha) = \frac{\alpha \lfloor W/2\pi \rfloor + \min(\alpha, W - 2\pi \lfloor W/2\pi \rfloor)}{W} \quad (B4)$$

For $\frac{W}{2\pi} \gg 1$, this distribution approaches to a uniform one.

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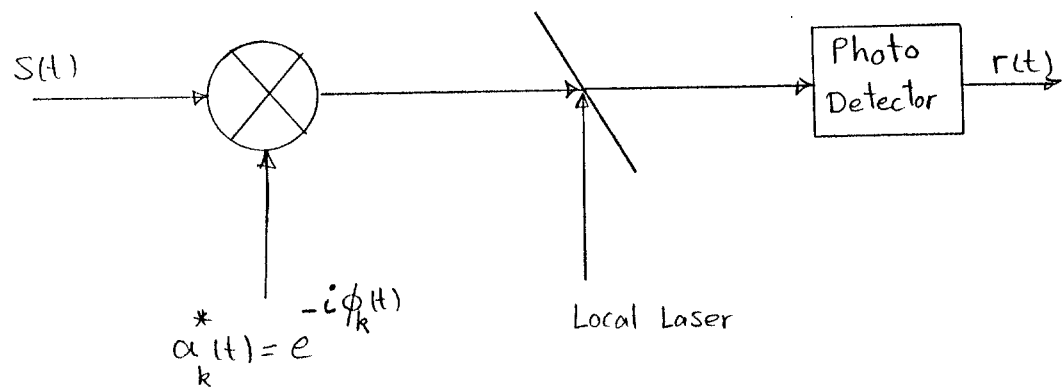


Fig.1.) Reception of the k -Th user. De-spreading followed by homodyne detection.

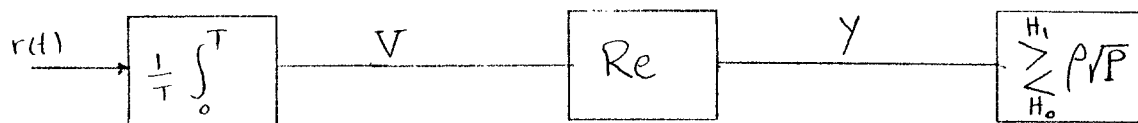


Fig.2.) Correlation receiver for the received electrical signal.